# Construction of Solutions to Electromagnetic Problems in Terms of Two Collinear Vector Potentials

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***Abstract* — In this work, the construction of solutions to general transient EM problems in terms of two collinear vector potentials (VPs) is subjected to a careful theoretical study and numerical verification. The problems are considered to be general in the sense that the medium can be inhomogeneous, lossy and may contain sources. Anisotropy is not considered in this paper. First, the completeness of the solution in terms of the two VPs is addressed. Second, the behavior of the VPs at interfaces and edges is investigated. Finally, a number of simple but relevant numerical tests are performed to verify the theoretical model. This work is part of the effort to establish the solid theoretical background of a novel efficient method for the analysis of transient EM propagation.**

## I. Introduction

It is well known that the electromagnetic (EM) field can be described not only in terms of the field vectors but also in terms of vector and scalar potentials. The following has been stated for time-harmonic fields but it is also true in the general transient case [1]: *"an arbitrary field in a homogeneous source-free region can be expressed as a sum of a TM field and a TE field"*. The TM (with respect to the direction of an arbitrary unit vector $\hat{c}$) field is described by the magnetic vector potential (VP), $\vec{A} = \hat{c}A$, and the TE field is the one of the electric VP $\vec{F} = \hat{c}F$. Both potentials are solutions to the wave equation in time domain (or the Helmholtz' equation in the frequency domain). Both vectors are collinear of fixed direction $\hat{c}$. Their magnitudes are sometimes referred to as wave potentials.

To the author's knowledge, the formulation of a solution to a general (lossy, inhomogeneous, involving sources) transient EM problem in terms of vector potentials has never been considered in detail probably because it was deemed to be too complicated for practical purposes. However, as it will be shown below, such analysis reveals interesting properties of the VPs, which make their implementation in practical numerical algorithms feasible and very promising.

This work has its roots in previous research, which resulted in the development of a time-domain algorithm based on the magnetic vector potential $\vec{A}$ and the second-order scalar wave equations of its three spatial components [2]. Another algorithm based on a single VP was reported in [3]. However, first applications of a pair of collinear VPs of fixed direction, which solved the wave equations for just two scalar wave potentials, were shown only recently [4].

These first applications in the form of a finite-difference algorithm revealed several problems, which needed further careful study. The choice of the direction of the VPs was crucial when dielectric interfaces were present, especially between regions, whose dielectric constants would differ significantly (by a factor of four and more). This choice was also important when corners and edges were present. It became obvious that the bottleneck is the formulation of the boundary conditions for the VPs. These observations made it imperative that a general model of the VP propagation is developed, which gives a clear picture of their behavior at material interfaces and inhomogeneities.

## II. General Vector Potential Equations

One starts with the introduction of the magnetic VP $\vec{A}$ and the electric VP $\vec{F}$ as:

$$\vec{H}^A = \frac{1}{\mu}\nabla\times\vec{A};\quad \vec{E}^F = -\frac{1}{\varepsilon}\nabla\times\vec{F} \tag{1}$$

Here, $\vec{H}^A$ is the magnetic field vector of a field associated with electric sources only ($\nabla\cdot\vec{B}^A = 0$). The $\vec{E}^F$ vector is the electric field associated with magnetic sources only ($\nabla\cdot\vec{D}^F = 0$). Their counterparts, $\vec{E}^A$ and $\vec{H}^F$, will be found by substitution in Maxwell's equations. The total field is a superposition of $(\vec{E}^A, \vec{H}^A)$ and $(\vec{E}^F, \vec{H}^F)$. Note that this implies linear media. The next step is to substitute (1) in Maxwell's equations and split them into two systems of equations as shown below.

$$\begin{cases} \nabla\times\vec{H}^F = -\nabla\times\dfrac{\partial\vec{F}}{\partial t} - \nabla\times\left(\dfrac{\sigma}{\varepsilon}\vec{F}\right) + \nabla\times\left[\left(\nabla\dfrac{1}{\mu}\right)\times\vec{A}\right] \\ \mu\dfrac{\partial\vec{H}^F}{\partial t} = \nabla\times\nabla\times\dfrac{\vec{F}}{\varepsilon} - \sigma_m\vec{H}^F + \left(\nabla\dfrac{\sigma_m}{\mu}\right)\times\vec{A} - \vec{M}^i \end{cases} \tag{2}$$

$$\begin{cases} \varepsilon \dfrac{\partial \vec{E}^A}{\partial t} = \nabla\times\nabla\times\dfrac{\vec{A}}{\mu} - \sigma\vec{E}^A - \left(\nabla\dfrac{\sigma}{\varepsilon}\right)\times\vec{F} - \vec{J}^i \\ \nabla\times\vec{E}^A = -\nabla\times\dfrac{\partial\vec{A}}{\partial t} - \nabla\times\left[\left(\nabla\dfrac{1}{\varepsilon}\right)\times\vec{F}\right] - \nabla\times\left(\dfrac{\sigma_m}{\mu}\vec{A}\right) \end{cases} \quad (3)$$

In the above transformations, the following identities have been used:

$$\begin{aligned} \nabla\times\left(\frac{1}{\mu}\nabla\times\vec{A}\right) &= \nabla\times\left[\nabla\times\left(\frac{\vec{A}}{\mu}\right) - \nabla\left(\frac{1}{\mu}\right)\times\vec{A}\right] \\ \nabla\times\left(\frac{1}{\varepsilon}\nabla\times\vec{F}\right) &= \nabla\times\left[\nabla\times\left(\frac{\vec{F}}{\varepsilon}\right) - \nabla\left(\frac{1}{\varepsilon}\right)\times\vec{F}\right] \end{aligned} \quad (4)$$

From the first equation in (2) it follows that

$$\vec{H}^F = -\frac{\partial\vec{F}}{\partial t} - \nabla\Psi + \left(\nabla\frac{1}{\mu}\right)\times\vec{A} - \frac{\sigma}{\varepsilon}\vec{F}, \quad (5)$$

and from the second equation in (3) it follows that

$$\vec{E}^A = -\frac{\partial\vec{A}}{\partial t} - \nabla\Phi - \left(\nabla\frac{1}{\varepsilon}\right)\times\vec{F} - \frac{\sigma_m}{\mu}\vec{A} \quad (6)$$

Note the cross-coupling between the *F*-field and the *A*-potential and the *A*-field and the *F*-potential due to the constants' non-zero gradients. This cross-coupling appears also in the governing equations of the VPs given below. If one applies Lorentz' gauge to the functions $\vec{A}/\mu$ and $\vec{F}/\varepsilon$ (instead of $\vec{A}$ and $\vec{F}$):

$$-\frac{1}{\varepsilon}\nabla\cdot\left(\frac{\vec{A}}{\mu}\right) = \frac{\partial\Phi}{\partial t} + \frac{\sigma}{\varepsilon}\Phi; \quad -\frac{1}{\mu}\nabla\cdot\left(\frac{\vec{F}}{\varepsilon}\right) = \frac{\partial\Psi}{\partial t} + \frac{\sigma_m}{\mu}\Psi, \quad (7)$$

one arrives at their general wave equations:

$$\begin{aligned} &\mu\varepsilon\frac{\partial^2}{\partial t^2}\left(\frac{\vec{A}}{\mu}\right) - \nabla^2\left(\frac{\vec{A}}{\mu}\right) + \left(\varepsilon\sigma_m + \mu\sigma\right)\frac{\partial}{\partial t}\left(\frac{\vec{A}}{\mu}\right) + \sigma\sigma_m\left(\frac{\vec{A}}{\mu}\right) \\ &-(\nabla\varepsilon)\frac{\partial\Phi}{\partial t} - (\nabla\sigma)\Phi - \nabla\varepsilon\times\frac{\partial}{\partial t}\left(\frac{\vec{F}}{\varepsilon}\right) - \nabla\sigma\times\left(\frac{\vec{F}}{\varepsilon}\right) = \vec{J}^i \end{aligned} \quad (8)$$

$$\begin{aligned} &\mu\varepsilon\frac{\partial^2}{\partial t^2}\left(\frac{\vec{F}}{\varepsilon}\right) - \nabla^2\left(\frac{\vec{F}}{\varepsilon}\right) + \left(\mu\sigma + \varepsilon\sigma_m\right)\frac{\partial}{\partial t}\left(\frac{\vec{F}}{\varepsilon}\right) + \sigma\sigma_m\left(\frac{\vec{F}}{\varepsilon}\right) \\ &-(\nabla\mu)\frac{\partial\Psi}{\partial t} - (\nabla\sigma_m)\Psi + \nabla\mu\times\frac{\partial}{\partial t}\left(\frac{\vec{A}}{\mu}\right) + \nabla\sigma_m\times\left(\frac{\vec{A}}{\mu}\right) = \vec{M}^i \end{aligned} \quad (9)$$

The following important conclusions can be made from equations (8) and (9). 1) Collinear VPs, which are normal to dielectric/magnetic/loss interfaces, are not mutually coupled. The scattering of the VP pair at an orthogonal interface can be fully described by two scalar quantities: the magnitudes of the VPs. However, if the VP pair has tangential components at an interface, they will be mutually coupled. They will be indirectly coupled to their normal components, too. One has to consider all six coupled components, which makes the problem too complicated. 2) A normal to an interface component of a VP will never give rise to a tangential component of its own. On the contrary, a tangential to a dielectric interface component $\vec{A}_\tau$ will generate a normal component $A_n$. Same holds for $\vec{F}$ at magnetic interfaces.

In addition, further analysis shows that the boundary conditions (BCs) at conducting edges of a pair of VPs, which are tangential to the edge, are well posed. Homogeneous Dirichlet condition is imposed on the magnetic VP, $A_\tau = 0$, regardless of the direction from which the edge is approached. A homogeneous Neumann condition is imposed on the electric VP, $\partial F_\tau / \partial n = 0$, where $\hat{n}$ is *any* direction normal to the edge. On the contrary, if a VP is orthogonal to the edge, its boundary conditions are ill posed. They do depend on the direction from which the edge is approached. For example, at an *x*-directed edge (or wedge), $\partial F_z / \partial y = 0$ when the observation point approaches the edge along the *y*-axis. However, when the observation point approaches the edge along the *z*-axis, the BC is a Dirichlet one: $F_z = 0$. Such ill-posed BCs degrade the performance of numerical algorithms based on finite discrete meshes.

To summarize, if one can keep the VPs normal to interfaces and tangential to edges and wedges, two scalar quantities (the magnitudes of two collinear VPs) will be sufficient to describe the total field behavior without having to take care of mode coupling. It is now obvious that in order to solve practical problems involving material interfaces, edges and corners, in a robust and simple manner, one cannot keep the direction of the VP pair constant in space.

## III. Mode Equivalence

The above conclusion makes it imperative to study the transitions between pairs of VPs. These transitions are possible and there are clear rules to carry them out in a homogeneous medium, at least in the case of mutually orthogonal VP pairs. Thus, the computational region can be divided into domains of constant direction of the VP pair, such that mode coupling is avoided. This mode coupling will be taken care of in an implicit manner by the transition equations at the mutual boundaries of the neighboring domains. The mode equivalence has been also studied in order to prove the completeness of the solution to an EM problem in terms of a pair of collinear VPs. However, this proof will not be considered here for the sake of brevity.

Assume that a pair $(A_x, F_x)$ is to be related to a $(A_y, F_y)$ pair or to a $(A_z, F_z)$ pair in a neighboring domain. This has to be done in such a way that the field components expressed in terms of the $(A_x, F_x)$ pair are the same as those expressed in terms of the orthogonal pair

TABLE I
SUMMARY OF MODE EQUIVALENCE FORMULAS IN HOMOGENEOUS REGIONS

| Field component | $(A_x, F_x)$ | $(A_y, F_y)$ | $(A_z, F_z)$ |
|---|---|---|---|
| $-E_x$ | $\partial_t A_x + \partial_x \Phi_x$ | $\partial_x \Phi_y - \partial_z F_y / \varepsilon$ | $\partial_x \Phi_z + \partial_y F_z / \varepsilon$ |
| $-E_y$ | $\partial_y \Phi_x + \partial_z F_x / \varepsilon$ | $\partial_t A_y + \partial_y \Phi_y$ | $\partial_y \Phi_z - \partial_x F_z / \varepsilon$ |
| $-E_z$ | $\partial_z \Phi_x - \partial_y F_x / \varepsilon$ | $\partial_z \Phi_y + \partial_x F_y / \varepsilon$ | $\partial_t A_z + \partial_z \Phi_z$ |
| $-H_x$ | $\partial_t F_x + \partial_x \Psi_x$ | $\partial_x \Psi_y + \partial_z A_y / \mu$ | $\partial_x \Psi_z - \partial_y A_z / \mu$ |
| $-H_y$ | $\partial_y \Psi_x - \partial_z A_x / \mu$ | $\partial_t F_y + \partial_y \Psi_y$ | $\partial_y \Psi_z + \partial_x A_z / \mu$ |
| $-H_z$ | $\partial_z \Psi_x + \partial_y A_x / \mu$ | $\partial_z \Psi_y - \partial_x A_y / \mu$ | $\partial_t F_z + \partial_z \Psi_z$ |

(either $(A_y, F_y)$ or $(A_z, F_z)$). The transition from one pair to another is done using the longitudinal field components, which depend on a single potential (see the highlighted formulas in the Table 1).

The introduction of sub-regions of constant direction of the VP pair eliminates the complications arising from mode coupling. It makes the proposed theoretical model efficient and applicable to the solution of practical EM problems. It is clearly simpler to implement in cases of predominantly homogeneous regions with planar interfaces such as layered structures.

## IV. FINITE-DIFFERENCE IMPLEMENTATION

Central differences are used throughout the algorithm based on the finite-difference discretization of the wave equations of the VPs and the transition formulas given in Table 1. Every two collinear VPs are displaced by half a step along all three axes (see Fig. 1). Besides, the magnetic VPs are displaced by half a step in time with respect to the electric VPs.

The boundary conditions at dielectric and magnetic interfaces are automatically satisfied by the VP functions, which are normal to these interfaces. Because of the continuity of the tangential $\vec{E}_\tau$ field and the continuity of the tangential $\vec{H}_\tau$ field, it can be shown that the functions:

$$\frac{A_n}{\mu},\ \frac{\partial}{\partial n}\left(\frac{A_n}{\mu}\right),\ \frac{F_n}{\varepsilon},\ \text{and}\ \frac{\partial}{\partial n}\left(\frac{F_n}{\varepsilon}\right),$$

are also continuous across dielectric and magnetic interfaces. Here, $\partial / \partial n$ denotes differentiation with respect to the interface unit normal $\hat{n}$. Therefore, the values of the VPs at the interfaces are calculated using the general equations in (8) and (9). The $F_n$ equation at dielectric interfaces and the $A_n$ equation at magnetic interfaces do not differ from their respective equations in points of zero $\varepsilon$ and $\mu$ gradients. The $A_n$ equation at dielectric interfaces differs from the respective zero-gradient equation by its modified $\nabla^2$ operator as dictated by (8). For example, if $\hat{n} \equiv \hat{x}$, then the Laplacian operator in the $A_n$ equation is replaced by:

$$\nabla^2\left(\frac{A_n}{\mu}\right) + (\nabla\varepsilon)\frac{\partial \Phi}{\partial t} = \nabla_\perp^2\left(\frac{A_n}{\mu}\right) + \varepsilon\frac{\partial}{\partial x}\left(\frac{1}{\varepsilon}\frac{\partial}{\partial x}\frac{A_n}{\mu}\right) \quad (10)$$

where $\nabla_\perp^2 = \partial_{yy}^2 + \partial_{zz}^2$. The equation for $F_n$ is dual to the $A_n$ equation.

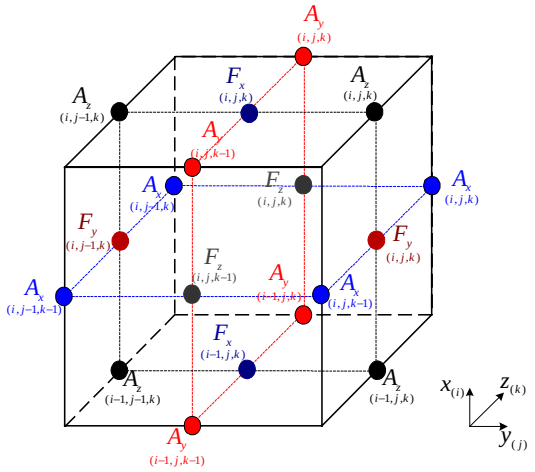


**Fig. 1.** A discretization cell showing the location in space of the three possible VP pairs in a rectangular coordinate system.

According to the number of operations per cell, in its finite-difference implementation, the algorithm would require 2/3 of the computation time and 2/3 of the memory requirements of the Yee-cell FDTD algorithm. Stability criteria, discretization cell size and excitation waveforms are chosen exactly as in the conventional FDTD algorithm.

## V. NUMERICAL TESTS

There are two important issues, which have been tested numerically. The first one is the transition between domains with different direction of the VP pair. The second one is the correct field representation at edges and corners.

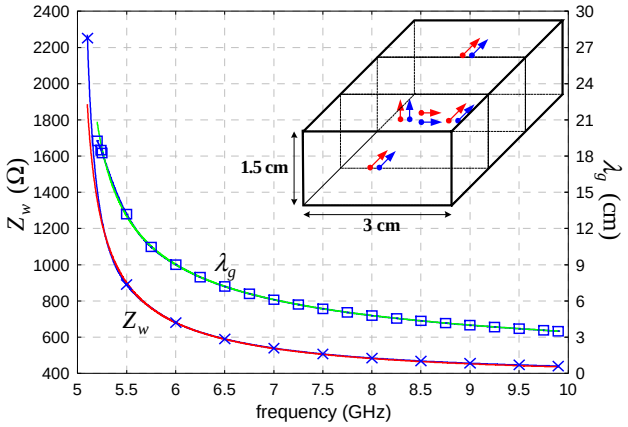


**Fig. 2.** Impedance and wavelength computations for a waveguide using a middle domain with three different VP pairs: with lines – analytical calculations; with line-points – numerical tests.

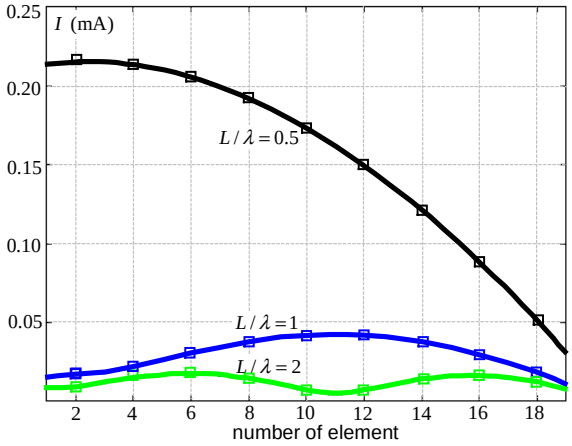


**Fig. 3.** Current distribution along half the length of a wire dipole: with lines – single potential; with points – computational region divided into 7 domains.

### *A. Transitions between domains*

The transition between VP pairs of different (orthogonal) directions has been tested in homogeneous problems since the domain boundaries in the general algorithm are always at least 2 cells away from inhomogeneities. One of the test structures is shown in Fig. 2. This is a hollow waveguide of rectangular cross-section excited with a dominant $TE_{01}$ distribution of the $E_x$ component whose waveform is a sine wave modulated by Blackman-Harris window. The input and output sections are the domains of the $(A_z, F_z)$ pair. The middle section supports: the $(A_z, F_z)$ pair in the first experiment, the $(A_y, F_y)$ pair in the second experiment, and the $(A_x, F_x)$ pair in the third experiment. In all three experiments, the wave impedance and the wavelength were calculated to compare with the analytical formulas. The results of all three experiments are practically indistinguishable from each other, and they deviate only slightly from the analytical calculations at low frequencies, close to cut-off (see Fig. 2).

A second test was performed on an open problem: an infinitesimally thin dipole antenna, where the middle domain centered onto the antenna has all its six walls bordering domains whose VPs are orthogonal to those in the middle domain. The current distribution (at different frequencies) is compared with the current distribution obtained when the problem is solved using only one potential: the magnetic VP component tangential to the dipole (see Fig. 3).

### *B. Dielectric interfaces, edges and corners*

These tests include: H-plane and E-plane waveguide bends, waveguide post, partially filled waveguide, microstrip line and fin-line.

## VI. Conclusion

In this work, the construction of solutions to general transient EM problems in terms of two collinear vector potentials (VPs) is considered. It is shown that as long as the gradient of the electromagnetic properties of the analyzed region coincides with the direction of the VPs, the solution can be built only on two scalar quantities: the magnitudes of the VPs. Such a restriction on the gradient of the material constants would severely limit the applicability of the method if VPs of fixed direction were used throughout the volume. However, it has been shown that there are no theoretical obstacles for the utilization of VPs of different directions in different sub-regions of the analyzed volume. The method is implemented in a finite-difference algorithm, which has better computational efficiency than the conventional FDTD technique and is comparable to it with respect to versatility and ease of implementation.

## References

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